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Scattering on a hyperbolic torus in a constant magnetic field

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Abstract. We study the quantum mechanical scattering of a particle on a two-dimensional non-compact Riemannian manifold of constant negative curvature embedded in a constant external magnetic field. The interplay between fundamental group operations and gauge transformations allows us to compute the scattering states. A closed expression for the phase shift of a plane wave entering the torus through its leak is shown to involve the quantized magnetic flux.

1. Introduction

Since the pioneering work by Hadamard in 1898 on the geodesic flow on a two-dimensional Riemannian manifold of constant negative curvature, the hyperbolic geometry has inspired both mathematicians and physicists [1]. In the field of dynamical systems, the first result was the proof by Hedlung and Hopf that the free motion (geodesic) on such a compact surface is ergodic (see, for instance, [2]). One now knows that this is a particular case of a Bernoulli system, the most chaotic type that can be found in Nature according to the ergodic hierarchy [3] (even if an old theorem due to Hilbert (see, for instance, [4]) states that such a surface cannot exist embedded in a three-dimensional Euclidean space).

More recently the quantum scattering of a particle on a two-dimensional manifold of genus one constructed by identifying the sides of a fundamental domain of the hyperbolic plane associated with a subgroup of the modular group $SL(2,Z)$ has been studied [5, 6]. This surface is topologically a torus with an infinite horn attached to it (therefore a torus with a leak). Using mathematical results of the scattering theory for automorphic functions, one can calculate the scattering states on such a manifold. The physical picture is that one injects a particle through the horn and then looks at what emerges. The scattering states are of the form

$$\psi_k = \psi_k^{\mathrm{in}} + \mathrm{e}^{\mathrm{i}\beta(k)}\psi_k^{\mathrm{out}}$$

where k is the wavenumber and the phase shift $\beta(k)$ is a real function involving the Riemann zeta function on the line $\mathrm{Re}\,(s) = 1$, defined for $\mathrm{Re}\,(s) > 1$ by

$$\zeta(s) \equiv \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

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and analytically continued elsewhere (the other part of the phase being a smooth regular function). Numerical studies of this phase shift as a function of the wavenumber k show that it exhibits a smooth 'random-like' behaviour. A rigorous result, the Reich–Voronin theorem [7], strengthens this chaotic behaviour due to the zeta function. It states that $\zeta(s)$ is a universal smooth function on the critical band $\frac{1}{2} < \mathrm{Re}(s) < 1$ in the sense that it can mimic an arbitrary analytic function as close as one wants in that band.

In the present work, we investigate the phase shift when the leaky torus is embedded in a constant magnetic field (in the sense of the hyperbolic measure). This paper is organized as follows. In section two, we determine the Killing vectors of the hyperbolic plane, which will turn out to be useful when we introduce the gauge potential. In section three we briefly review the classical motion of a particle on the hyperbolic plane embedded in a constant magnetic field, following earlier work of one of us [8]. In the fourth section, we establish the link between gauge transformations and identifications of the sides of the fundamental domain by using the Lie derivative formalism. We finally calculate the scattering states in the magnetic field. The non-trivial topology of the manifold implies that the magnetic field must be quantized. (The scalar curvature R of the manifold introduces a length scale $1/\sqrt{|R|}$ and thus a magnetic strength scale $B_0 = \hbar|R|/e$. A quantized field is a field whose strength B is an integer $n \in Z$ in the unit scale B_0.) It is shown that the new scattering states again take the form

$$\tilde{\psi}_k = \psi_k^{\mathrm{in}} + \mathrm{e}^{\mathrm{i}\delta(k;B)}\psi_k^{\mathrm{out}}$$

where the Riemann zeta function appears in δ in the same way as in β, and where the effects of B only show up in its smooth regular part.

2. Killing vectors and the hyperbolic plane

It is well known that any continuous geometrical symmetry associated with a Riemannian manifold M can be described by the Lie derivative formalism of differential geometry [9]. The problem of finding the symmetry group of a Riemannian manifold described by its metric tensor $g_{\mu\nu}$ is equivalent to that of finding the set of all independent Killing vector fields ξ_j $\{j = 1, ..., k\}$ on the tangent bundle of the manifold. They are solutions of the partial differential equations [10]

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \xi^\lambda + g_{\mu\lambda}\partial_\nu \xi^\lambda = 0 \tag{2.1}$$

where $\mathcal{L}_\xi g$ is the Lie derivative of the metric tensor field g. The integral curves are generated by the vector field ξ through the one-parameter group of motion

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = \xi^\mu(x(t)). \tag{2.2}$$

Equation (2.1) can be rewritten by using the covariant derivative on M as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{2.3}$$

where

$$\nabla_\mu X_\nu \equiv \partial_\mu X_\nu - \Gamma_\mu{}^\rho{}_\nu X_\rho.$$

The Killing vectors completely determine the group of continuous isometry of the manifold, namely the set of transformations which preserves the Riemannian length and the angles. In physical terms, they determine the symmetry group of a non-relativistic free particle of mass m lying on that manifold since its Lagrangian is $L = \frac{1}{2} m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$.

Let us derive the symmetry group of the hyperbolic plane, namely $SL(2,R)/Z_2$. In the model of the Poincaré upper half plane $H = \{z \equiv x + \mathrm{i}y,\ y > 0\}$, the Riemannian line element is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{2.4}$$

where we have normalized the Gauss curvature to be -1. The Killing equations are

$$\partial_x \xi^y + \partial_y \xi^x = 0 \qquad \partial_x \xi^x = \frac{\xi^y}{y} \qquad \partial_y \xi^y = \frac{\xi^y}{y} \tag{2.5}$$

with the general solution

$$\xi^x = 3a(y^2 - x^2) - 2bx - c \qquad \xi^y = -y(6ax + 2b) \tag{2.6}$$

where a, b and c are three arbitrary constants. One thus has three independent Killing vector fields, which can be chosen as (up to an overall normalization constant)

$$\xi_1^\mu = \begin{cases} 1 \\ 0 \end{cases} \tag{2.7}$$

associated through equation (2.2) with the translations along the x axis, namely $z(t) = z_0 + t$;

$$\xi_2^\mu = \begin{cases} x \\ y \end{cases} \tag{2.8}$$

associated with uniform dilatation $z(t) = \mathrm{e}^t z_0$, and

$$\xi_3^\mu = \begin{cases} y^2 - x^2 \\ -2xy \end{cases} \tag{2.9}$$

associated with nonlinear rotation $z(t) = z_0/(1 + z_0 t)$.

Since the ξ form a Lie algebra, a general isometry will be generated by a linear combination $\xi \equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3$ where the α_i are real numbers, leading through (2.2) to the well known compact form of a fractional linear transformation (since the overall normalization of ξ can be absorbed in a redefinition of time, they are only three independent parameters)

$$z(t) = \frac{a(t) z_0 + b(t)}{c(t) z_0 + d(t)} \tag{2.10}$$

where

$$a(t) = \cosh\left(\frac{\sigma t}{2}\right) + \frac{\alpha_2}{\sigma} \sinh\left(\frac{\sigma t}{2}\right) \tag{2.11a}$$

$$b(t) = \frac{2\alpha_1}{\sigma} \sinh\left(\frac{\sigma t}{2}\right) \tag{2.11b}$$

$$c(t) = \frac{2\alpha_3}{\sigma} \sinh\left(\frac{\sigma t}{2}\right) \tag{2.11c}$$

$$d(t) = \cosh\left(\frac{\sigma t}{2}\right) - \frac{\alpha_2}{\sigma} \sinh\left(\frac{\sigma t}{2}\right) \tag{2.11d}$$

with $\sigma \equiv \sqrt{4\alpha_1\alpha_3 + \alpha_2^2}$ and the normalization $ad - bc \equiv 1$ for all t. There is an homomorphism between the algebra of the 2×2 matrices

$$\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.12}$$

normalized by $\det \gamma = 1$, and these fractional linear transformations. The Killing vectors of the hyperbolic plane will be useful when we will discuss gauge transformations on the leaky torus.

3. Classical dynamics on the hyperbolic plane in a constant magnetic field

The free motion on the hyperbolic plane takes place on the geodesics of this manifold, solutions of the dynamical equations

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma_\nu{}^\mu{}_\rho \frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = 0 \tag{3.1}$$

These are half circles whose centre are on the real axis $y = 0$, including vertical half straight lines as degenerate cases. The introduction of a vector potential A_μ with $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ gives the new dynamical equations of motion

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma_\nu{}^\mu{}_\rho \frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = \frac{e}{m} F^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}t} \tag{3.2}$$

where m is the mass of the particle and e its charge. A constant magnetic field over the manifold (in the sense of the metric) is a field whose strength tensor covariant derivative vanishes everywhere, i.e.

$$\nabla_\mu F_{\nu\rho} = 0. \tag{3.3}$$

Equivalently in two space dimensions, this means that the 2-form $B = \mathrm{d}A$ is proportional to the volume form

$$\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}. \tag{3.4}$$

In order to describe the trajectories in the presence of the magnetic field, we introduce the tangent vector to the geodesic parametrized by the geodesic length s

$$\alpha^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{3.5}$$

and the covariant derivative along the geodesic by

$$\nabla_s \equiv \alpha^\mu \nabla_\mu. \tag{3.6}$$

The covariant Serret–Frenet equations in two dimensions take the form

$$\nabla_s \alpha^\mu = \kappa\beta^\mu \qquad \nabla_s \beta^\mu = -\kappa\alpha^\mu \tag{3.7}$$

where κ is the intrinsic curvature of the trajectory and α^μ and β^μ form the local orthonormal Serret–Frenet basis. The dynamical equations (3.2) may be rewritten as

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\alpha^\mu + \kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \beta^\mu = \frac{e}{m}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) F^\mu{}_\nu \alpha^\nu . \tag{3.8}$$

By multiplying this equation by α_μ and taking into account the antisymmetry of the field strength tensor and orthonormality of the basis vectors, one finds that the trajectory is spanned at a constant speed $v_0 \equiv \mathrm{d}s/\mathrm{d}t$. Equation (3.8) then takes the form

$$\kappa v_0 \beta^\mu = \frac{e}{m} F^\mu{}_\nu \alpha^\nu . \tag{3.9}$$

Operating with ∇_s on (3.9) and then multiplying by β_μ gives

$$\frac{\mathrm{d}\kappa}{\mathrm{d}s} v_0 = \frac{e}{m}\left(\nabla_s F^\mu{}_\nu\right)\alpha^\nu \beta_\mu . \tag{3.10}$$

Thus, if the field strength tensor has a vanishing covariant derivative, the intrinsic curvature of the trajectory κ is a constant. (For a three-dimensional manifold, similar arguments would have led to a curve which, moreover, possesses constant torsion. See also [11].) The local basis thus satisfies the constraint

$$F_{\mu\nu}\beta^\nu \alpha^\mu = -\frac{\kappa m v_0}{e} \tag{3.11}$$

and, as shown in [8], the trajectories on the Poincaré upper half plane are still circles (in the usual sense since the hyperbolic space is conformal), but their centres do not automatically lie on the horizontal axis, depending on the strength of the magnetic field. For a value of the field greater than some critical value (depending on the energy), the particle gets trapped on closed orbits, being otherwise scattered on a half circle. It must be noticed that this behaviour is very different from the flat space case, where scattering trajectories disappear for arbitrary small positive value of the magnetic field.

4. Group operations of the leaky torus and gauge transformations

The leaky torus is constructed by the identification of the sides of a fundamental domain D of the hyperbolic plane in a way analogous to the flat space case [12]. The boundaries of the domain D in the Poincaré upper half plane are the four geodesics

$$x = -1 \qquad 0 < y < \infty \tag{4.1a}$$

$$(x + \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4} \qquad -1 < x < 0 \tag{4.1b}$$

$$x = +1 \qquad 0 < y < \infty \tag{4.1c}$$

$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4} \qquad 0 < x < 1 \tag{4.1d}$$

and identifications are made with the two fundamental operations A and B

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} . \tag{4.2}$$

A maps the boundary (4.1a) into (4.1d) and B maps (4.1c) into (4.1b) (see figure 1). The hyperbolic plane is tessellated by an infinite number of copies of the fundamental domain D generated by the application on D of some unique word constructed from A, B, A^{-1} and B^{-1} as long as the 'single/return' operations (namely AA^{-1}, $A^{-1}A$, BB^{-1}, $B^{-1}B$) are omitted. The set of all these transformations forms a discrete group Γ, a subgroup of the modular group $SL(2, Z)$ generated by A and B. Since the domain D is non-compact, identifications of its edges leaves a unique point infinitely far removed and the non-compact (but of finite area) manifold has thus the topology of a torus with an infinitely thin horn attached to it. This point at infinity is the only cusp of Γ whose stabilizer Γ_∞ is generated by the parabolic element $K \equiv B^{-1}A^{-1}BA$.

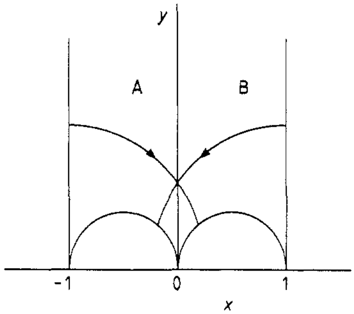

Figure 1. The fundamental domain associated with the subgroup Γ of the modular group $SL(2, Z)$ on the Poincaré upper half plane and the action of the generators A and B leading to the leaky torus by the gluing of the boundaries.

For a covariant constant magnetic field B, a particular gauge choice leads to

$$A_x = -\frac{B}{y} \qquad A_y = 0. \tag{4.3}$$

Now the gauge potential has to match up to a gauge transformation when one identifies the sides of D by the mean of the group generators A and B. These transformations are some combinations of the three independent motions associated with Killing vectors of the hyperbolic plane, thus it is sufficient to restrict the investigation to these motions. One can show that

$$(\mathcal{L}_{\xi_j}A)_\mu = 0 \tag{4.4}$$

for $j = 1$ and 2. On the contrary, for nonlinear rotations, one finds that

$$(\mathcal{L}_{\xi_3}A)_\mu = \left\{ \begin{array}{l} 0 \\ 2B \end{array} \right. = \partial_\mu \varphi(x, y). \tag{4.5}$$

This is a weak symmetry condition [13] since the gauge potential is only turned by a gauge transformation when one follows the curve generated by ξ_3. However, one wants an explicit matching of the gauge potential through the identifications, namely a gauge transformation such that the potential at the transformed point takes the value of the former one at the initial point. Writing this new potential as $A'_\mu = A_\mu - \partial_\mu \Lambda$, the equation $(\mathcal{L}_\xi A')_\mu = 0$ where $\xi \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ implies that the local rate of change of Λ is $\mathcal{L}_\xi \Lambda = \alpha_3 \varphi$. For a finite transformation γ generated by ξ in the finite time t , the point z is transformed into γz. Taking z as a reference point by setting $\Lambda(z) \equiv 0$, the value of Λ at the point γz is given by

$$\Lambda(\gamma z) = \int_0^{\Lambda(\gamma z)} \mathrm{d}\Lambda = \int_z^{\gamma z} \mathrm{d}x^\mu \, \partial_\mu \Lambda = \int_0^t \mathrm{d}\tau \, \mathcal{L}_\xi \Lambda \tag{4.6}$$

where (2.2) has been used. Thus

$$\Lambda(\gamma z) = \alpha_3 \, 2B \int_0^t \mathrm{d}\tau \; y(\tau) \tag{4.7}$$

where

$$y(\tau) = \frac{y}{|c(\tau)z + d(\tau)|^2}. \tag{4.8}$$

The integral is carried out with the help of (2.11c) and (2.11d), leading to

$$\Lambda(\gamma z) = 2B \tan^{-1}\left(\frac{y}{x + d/c}\right) = 2B \arg(cz + d). \tag{4.9}$$

5. Scattering states

The calculation of the scattering states begins with the construction of the Hamiltonian. The modified Laplace–Beltrami operator acting on scalar functions is given by

$$\begin{aligned} \tilde{\Delta} &\equiv -g^{\mu\nu}(\nabla_\mu - \mathrm{i}eA_\mu)(\partial_\nu - \mathrm{i}eA_\nu) \\ &= -\Delta + 2\mathrm{i}eA^\mu\partial_\mu + \mathrm{i}e(\nabla_\mu A^\mu) + e^2 A^\mu A_\mu \end{aligned} \tag{5.1}$$

where Δ is the ordinary Laplace–Beltrami operator. The gauge choice (4.3) leads to the time-independent Schrödinger equation

$$H\psi(x,y) \equiv \left(-\frac{\hbar^2}{2m}\, y^2(\partial_x^2 + \partial_y^2) - \mathrm{i}\hbar\frac{eB}{m}\, y\partial_x + \frac{\mathrm{e}^2B^2}{2m}\right)\psi(x,y) = \lambda\psi(x,y). \tag{5.2}$$

A particular incident plane wave coming from $y = \infty$ and going down to $y = 0$

$$\psi_k(x,y) = y^{(1/2)-\mathrm{i}k} = [\mathrm{Im}\,(z)]^{(1/2)-\mathrm{i}k} \tag{5.3}$$

is a solution of the free scattering on the leaky torus and is also an eigenfunction of the Schrödinger equation (5.2), but the energy λ and the wavenumber k are now related by

$$\lambda = \frac{\hbar^2}{2m}\left[\frac{1}{4} + k^2 + \left(\frac{eB}{\hbar}\right)^2\right]. \tag{5.4}$$

We have seen that under the operation $\gamma \in \Gamma$, $z \mapsto \gamma z$, the potential just changes by a pure gauge. It follows that the resulting wavefunction has to be gauge transformed accordingly, namely

$$\begin{aligned}\psi_k(z) = [\mathrm{Im}\,(z)]^{(1/2)-\mathrm{i}k} \mapsto (T_\gamma \psi_k)(z) &= \psi_k(\gamma z)\ \mathrm{e}^{\mathrm{i}(e/\hbar)\Lambda(\gamma z)} \\ &= \frac{y^{(1/2)-\mathrm{i}k}}{|cz+d|^{1-2\mathrm{i}k}} \left(\frac{cz+d}{c\bar{z}+d}\right)^{eB/\hbar}.\end{aligned} \tag{5.5}$$

One can verify directly that the latter expression is indeed an eigenfunction of the Hamiltonian with the same energy.

The full wavefunction is then obtained by summing over all the group elements modulo the right multiplications by K since this generates translations of six units along the x axis, namely $K^n \gamma z = \gamma z + 6n$, leaving c and d unaffected. So one sums over the right cosets $\Gamma_\infty \backslash \Gamma$ of Γ with respect to K in order to avoid infinite repetition and gets the generalized automorphic function [14]

$$\Psi_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_k(\gamma z) \mathrm{e}^{\mathrm{i}(e/\hbar)\Lambda(\gamma z)}. \tag{5.6}$$

Let us now show that the magnetic field has to be quantized, just as it is on a flat torus. For this, we consider the wavefunction at the four corners of the fundamental domain. Since these points are the same through identifications of the edges, the wavefunction must take the same value up to a gauge transformation at each of these. One thus obtains

$$A(-1,+\infty) = (+1,0) \Rightarrow \psi(+1,0) = \mathrm{e}^{+\mathrm{i}\pi eB/\hbar}\psi(-1,+\infty) \tag{5.7a}$$

$$A(-1,0) = (0,0) \Rightarrow \psi(0,0) = \psi(-1,0) \tag{5.7b}$$

$$B(+1,+\infty) = (-1,0) \Rightarrow \psi(-1,0) = \mathrm{e}^{-\mathrm{i}\pi eB/\hbar}\psi(+1,+\infty) \tag{5.7c}$$

$$B(+1,0) = (0,0) \Rightarrow \psi(0,0) = \psi(+1,0). \tag{5.7d}$$

Now $\psi(x, y)$ is independent of x as y goes to infinity and so $\psi(-1,+\infty) = \psi(+1,+\infty)$. (Intuitively, the horizontal hyperbolic measure of the domain goes to zero as y goes to infinity so that the band becomes infinitely narrow and thus the problem one dimensional.) It then follows that

$$\psi(0,0) = \mathrm{e}^{+\mathrm{i}\pi eB/\hbar}\psi(+1,+\infty) \tag{5.8a}$$

$$\psi(0,0) = \mathrm{e}^{-\mathrm{i}\pi eB/\hbar}\psi(+1,+\infty) \tag{5.8b}$$

which leads to the quantization of the magnetic field strength, namely $B = n\hbar/e$ where $n \in Z$.

In order to evaluate the sum over the cosets, we just follow the calculation made in [5] according to the general pattern of [14, 15]. One first takes advantage of the periodicity of $\Psi_k(z)$ in x by performing a Fourier expansion in this variable. To visualize the invariance under right multiplications by K , it is convenient to redesign the fundamental domain D as a juxtaposition of six modular domains [5] lying in the band $x \in [-\frac{5}{2}, +\frac{7}{2}]$. This gives

$$\Psi_k(z) = \sum_{m \in Z} a_{k,m}(y)\ \mathrm{e}^{+2\pi\mathrm{i}mx/6} \tag{5.9a}$$

where

$$a_{k,m}(y) = \frac{1}{6}\int_{-5/2}^{+7/2} \mathrm{d}x\, \mathrm{e}^{-2\pi i m x/6}\Psi_k(x,y). \tag{5.9b}$$

The next step is to transform the sum over the cosets of Γ into a sum over prime numbers (the normalization condition on the coefficients of a transformation γ tells that c and d are relatively prime, which is denoted by $(c,d)=1$), yielding

$$\begin{aligned} a_{k,m}(y) = y^{(1/2)-\mathrm{i}k}\,\delta_{m,0} + \frac{1}{6}\sum_{c=1}^{+\infty}\sum_{\substack{-\infty\leq d\leq+\infty\\(c,d)=1}} \int_{-5/2}^{+7/2}\mathrm{d}x\,\frac{y^{(1/2)-\mathrm{i}k}}{|cz+d|^{1-2\mathrm{i}k}} \\ \times\left(\frac{cz+d}{c\bar z+d}\right)^{eB/\hbar}\mathrm{e}^{-2\pi\mathrm{i}mx/6} \\ = y^{(1/2)-\mathrm{i}k}\delta_{m,0} + \frac{1}{6}\sum_{c=1}^{+\infty}\sum_{\substack{0<d<6c\\(c,d)=1}}\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{y^{(1/2)-\mathrm{i}k}}{|cz+d|^{1-2\mathrm{i}k}} \\ \times\left(\frac{cz+d}{c\bar z+d}\right)^{eB/\hbar}\mathrm{e}^{-2\pi\mathrm{i}mx/6}. \end{aligned} \tag{5.10}$$

For the zero Fourier coefficient, one finds with the help of [16],

$$\begin{aligned} \int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{y^{1/2-\mathrm{i}k}}{|cz+d|^{1-2\mathrm{i}k}}\left(\frac{cz+d}{c\bar z+d}\right)^{eB/\hbar} \\ = -\pi\,\frac{y^{(1/2)+\mathrm{i}k}}{c^{1-2\mathrm{i}k}}\,\frac{2^{1+2\mathrm{i}k}}{2\mathrm{i}k}\,\frac{\Gamma(1-2\mathrm{i}k)}{\Gamma(\frac{1}{2}-\mathrm{i}k+eB/\hbar)\Gamma(\frac{1}{2}-\mathrm{i}k-eB/\hbar)} \end{aligned} \tag{5.11}$$

where $\Gamma(x)$ is the Euler function such that

$$\begin{aligned} a_{k,0}(y) = y^{(1/2)-\mathrm{i}k} - \pi\,y^{(1/2)+\mathrm{i}k}\,\frac{2^{1+2\mathrm{i}k}}{2\mathrm{i}k}\,\frac{\Gamma(1-2\mathrm{i}k)}{\Gamma(\frac{1}{2}-\mathrm{i}k+eB/\hbar)\Gamma(\frac{1}{2}-\mathrm{i}k-eB/\hbar)} \\ \times\sum_{c=1}^{+\infty}\sum_{\substack{0<d<c\\(c,d)=1}}\frac{1}{c^{1-2\mathrm{i}k}}. \end{aligned} \tag{5.12}$$

Using the number theory theorem [17]

$$\sum_{c=1}^{+\infty}\sum_{\substack{0<d<c\\(c,d)=1}}\frac{1}{c^{1-2\mathrm{i}k}} = \frac{\zeta(-2\mathrm{i}k)}{\zeta(1-2\mathrm{i}k)} \tag{5.13}$$

and some functional relations between the Euler function and the Riemann zeta function [16], one finally finds

$$a_{k,0}(y) = y^{1/2-\mathrm{i}k} + \pi^{-2\mathrm{i}k}\,\frac{\Gamma\left(\frac{1}{2}-\mathrm{i}k\right)\Gamma\left(\frac{1}{2}+\mathrm{i}k\right)}{\Gamma\left(\frac{1}{2}-\mathrm{i}k+eB/\hbar\right)\Gamma\left(\frac{1}{2}-\mathrm{i}k-eB/\hbar\right)}\,\frac{\zeta(1+2\mathrm{i}k)}{\zeta(1-2\mathrm{i}k)}\,y^{(1/2)+\mathrm{i}k}. \tag{5.14}$$

On the other hand, the space dependence of the non-zero Fourier coefficients is given by

$$a_{k,m}(y) \sim \begin{cases} W_{-eB/\hbar,\mathrm{i}k}\left(\dfrac{2\pi m y}{3}\right) & m > 0 \qquad (5.15a) \\ W_{+eB/\hbar,\mathrm{i}k}\left(-\dfrac{2\pi m y}{3}\right) & m < 0 \qquad (5.15b) \end{cases}$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function [18] whose asymptotic expansion is

$$W_{\alpha,\beta}(z) = \mathrm{e}^{-z/2}\, z^{\alpha}(1 + \mathrm{O}(1/z)) \qquad |z| \to +\infty. \tag{5.16}$$

Thus the non-zero Fourier coefficients vanish exponentially as $y \to +\infty$ and therefore do not contribute to the scattering. In other words, the asymptotic scattering states are given by the 's-wave' around the horn. We finally end up with

$$\psi_k(y) \equiv a_{k,0}(y) = \psi_k^{\mathrm{in}}(y) + \mathrm{e}^{\mathrm{i}\delta(k,B)}\psi_k^{\mathrm{out}}(y) \tag{5.17}$$

for the scattering states in the quantized constant magnetic field. The reflection coefficient $r \equiv \mathrm{e}^{\mathrm{i}\delta}$ is a pure phase

$$\begin{aligned} |r|^2 &= \frac{|\Gamma\left(\frac{1}{2} - \mathrm{i}k\right)\Gamma\left(\frac{1}{2} + \mathrm{i}k\right)|^2}{\Gamma\left(\frac{1}{2} - \mathrm{i}k + eB/\hbar\right)\Gamma\left(\frac{1}{2} - \mathrm{i}k - eB/\hbar\right)\Gamma\left(\frac{1}{2} + \mathrm{i}k + eB/\hbar\right)\Gamma\left(\frac{1}{2} + \mathrm{i}k - eB/\hbar\right)} \\ &= 1 - \frac{\sin^2(\pi eB/\hbar)}{\cosh^2(\pi k)} = 1 \end{aligned} \tag{5.18}$$

for a quantized magnetic field.

6. Conclusion

The final result

$$\mathrm{e}^{\mathrm{i}\delta(k,n)} = \pi^{-2\mathrm{i}k}\, \frac{\Gamma\left(\frac{1}{2} - \mathrm{i}k\right)\Gamma\left(\frac{1}{2} + \mathrm{i}k\right)}{\Gamma\left(\frac{1}{2} - \mathrm{i}k + n\right)\Gamma\left(\frac{1}{2} - \mathrm{i}k - n\right)}\, \frac{\zeta(1 + 2\mathrm{i}k)}{\zeta(1 - 2\mathrm{i}k)}$$

leads to a phase shift $\delta(k,n)$ that only depends of the magnetic field ($B = n\hbar/e$) through the Γ function. The chaotic properties of the phase shift discussed in detail in [5] are thus not affected by the presence of the field (only the non-fluctuating part of the phase shift is altered by the field). It is interesting to point out that the scattering amplitude displays complex poles (in the complex energy plane) associated with the non-trivial zeros of the ζ function and also poles on the real axis that only occur when the magnetic field is present. The latter are associated with the n bound states that are allowed in the presence of a quantized magnetic flux [8]. It is also interesting to discuss the scattering problem on the whole upper-half plane. In this case, the scattering states with a sharp momentum p along x read [8] (up to an overall normalization factor)

$$\psi = \mathrm{e}^{\mathrm{i}px}\, W_{-n,\mathrm{i}k}(2|p|y) \qquad p < 0.$$

In contrast with (5.9*a*), p is not quantized in the present case. These states decay exponentially for $y \to +\infty$ and display, at infinity, plane-wave behaviour. Indeed in the vicinity of the $y = 0$ axis

$$\psi \simeq e^{ipx} \left(\frac{\Gamma(-2ik)}{\Gamma(\frac{1}{2} - ik + n)} (2|p|y)^{(1/2)+ik} + \frac{\Gamma(2ik)}{\Gamma(\frac{1}{2} + ik + n)} (2|p|y)^{(1/2)-ik} \right)$$

describes for fixed p the superposition of an incoming and an outgoing plane wave along the y axis. The occurrence of such quasi-free states can be understood on a physical basis since, in this asymptotic region, the curvature effect largely dominates over the magnetic field. The corresponding phase shift reads

$$e^{i\delta_\infty} = \frac{\Gamma(-2ik)}{\Gamma(2ik)} \frac{\Gamma(\frac{1}{2} + ik + n)}{\Gamma(\frac{1}{2} - ik + n)} [2|p|]^{2ik}$$

to be compared with our final result for the phase shift on the hyperbolic torus (rewritten in a sligthly different form)

$$e^{i\delta(k,n)} = (-1)^n \pi^{-2ik} \frac{\Gamma(\frac{1}{2} + ik + n)}{\Gamma(\frac{1}{2} - ik + n)} \frac{\zeta(1 + 2ik)}{\zeta(1 - 2ik)}.$$

The dependence on the magnetic field is for both topologies given by a ratio of two gamma functions, although the wavefunctions are very different. A physical explanation of this striking similarity is still lacking.

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